Last name	First name	ID	Signature

Problem 1. A power-law fluid (with constant fluid properties, ρ, m, n) flows through a tapered tube with circular cross-section, for which the tube radius changes linearly from R_0 at the tube entrance to a slightly smaller value R_L at the tube exit.

1. Applying locally the result obtained for a circular tube (eq. 8.3-9), write the proper differential equation in $\mathrm{d}\mathcal{P}/\mathrm{d}z$;
2. Integrate the equation obtained in 1. in order to get the pressure gradient;
3. Compare the mass flow rate obtainable for the power-law fluid and for an incompressible fluid of constant properties (ρ, μ), flowing in the same tube in presence of the same pressure gradient (the solution for incompressible Newtonian fluid is given by eq. 2B.10-3).

Data. $L = 10$ m, $\rho = 1000$ kg/m^3, $m = 0.2$ Pa·s^n, $n = 0.85$, $R_0 = 10$ cm, $R_L = 9$ cm, $\Delta\mathcal{P}/L = 1.4$ bar/m, $\mu = 0.2$ Pa·s.

Problem 2. A piece of foodstuff shaped as a long cylinder with radius R and constant properties (k, $\rho\hat{C}_P$) was taken from a fridge with an initial uniform temperature T_0, and it was heated by hot air at temperature T_∞, flowing orthogonally to the cylinder axis, with velocity v_∞. Under these conditions, the Churchill & Bernstein correlation holds, and the interphase heat transfer coefficient has the value h. After a time t^*, the axial temperature of the foodstuff was measured to be T_A. Calculate:

1. the value of air velocity, v_∞;
2. the foodstuff thermal conductivity, k;
3. the surface temperature after the time t^*, $T_s(t^*, r = R)$.

The Churchill and Bernstein correlation is (D being the cylinder diameter):

$$N_{Nu} = \frac{hD}{k} = 0.3 + \frac{0.62 N_{Re}^{0.5} N_{Pr}^{0.33}}{\left[1 + \left(\frac{0.4}{N_{Pr}}\right)^{0.67}\right]^{0.25}} \left[1 + \left(\frac{N_{Re}}{282000}\right)^{0.625}\right]^{0.8}$$

The air properties could be taken as constants on their initial values.

Data. $R = 5$ cm, $\rho\hat{C}_P = 6.0$ MJ/(m^{3}K), $T_0 = 5$°C, $T_\infty = 150$°C, $h = 35$ W/(m^{2}K), $t^* = 0.5$ hr, $T_A = 30$°C.

Instructions: fill out the table above in this sheet. Use only this sheet for answers.

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$L := 10 \cdot m \quad \rho := 1000 \cdot \frac{kg}{m^3} \quad m := 0.2 \cdot Pa \cdot s^{0.85} \quad n := 0.85 \quad R_0 := 10 \cdot cm \quad R_L := 9 \cdot cm \quad \Delta P\_L := 1.4 \cdot \frac{Pa}{m} \quad \mu := 0.2 \cdot Pa \cdot s$

We consider a small region of the tapered tube to be a straight tube over a short distance dz; then we can write "locally"

$$w = \frac{\pi R^3 \rho}{(1/n)+3}\left[-\frac{d\mathcal{P}}{dz}\frac{R}{2m}\right]^{1/n} \qquad \text{eq. (8.3-9)}$$

$$dR\_dz := \frac{R_L - R_0}{L} = -1 \times 10^{-3}$$

Take the nth power of both sides to get

$$-\frac{d\mathcal{P}}{dz} = \frac{2m}{R}\left[\frac{w}{\pi R^3 \rho}\left(\frac{1}{n}+3\right)\right]^n$$

$$\frac{d}{dz}P = -\frac{2 \cdot m}{R} \cdot \left[\frac{w}{\pi \cdot R^3 \cdot \rho} \cdot \left(\frac{1}{n}+3\right)\right]^n$$

in which R is a function of z:

$$R = R_0 + \left(\frac{R_L - R_0}{L}\right)z$$

It is easier to integrate the differential equation if we rewrite it as

$$-\frac{d\mathcal{P}}{dR}\frac{dR}{dz} = -\frac{d\mathcal{P}}{dR}\left(\frac{R_L - R_0}{L}\right) = \frac{2m}{R}\left[\frac{w}{\pi R^3 \rho}\left(\frac{1}{n}+3\right)\right]^n$$

Then when this equation is integrated with respect to R, we get

$$-\int_{\mathcal{P}_0}^{\mathcal{P}_L} d\mathcal{P} = \left(\frac{2mL}{R_L - R_0}\right)\left[\frac{w}{\pi\rho}\left(\frac{1}{n}+3\right)\right]^n \int_{R_0}^{R_L}\frac{1}{R^{3n+1}}dR$$

Therefore

$$\mathcal{P}_0 - \mathcal{P}_L = \left(\frac{2mL}{R_L - R_0}\right)\left[\frac{w}{\pi\rho}\left(\frac{1}{n}+3\right)\right]^n\left(\frac{R_L^{-3n} - R_0^{-3n}}{-3n}\right)$$
$$= \left(\frac{2mL}{3n}\right)\left[\frac{w}{\pi\rho}\left(\frac{1}{n}+3\right)\right]^n\left(\frac{R_L^{-3n} - R_0^{-3n}}{R_0 - R_L}\right)$$

$$\frac{\Delta P}{L} = \frac{2 \cdot m}{3 \cdot n} \cdot \left[\frac{w}{\pi \cdot \rho} \cdot \left(\frac{1}{n}+3\right)\right]^n \cdot \left(\frac{R_L^{-3 \cdot n} - R_0^{-3 \cdot n}}{R_0 - R_L}\right)$$

$$w := \frac{\pi \cdot \rho}{\frac{1}{n}+3} \cdot \left[\frac{3 \cdot n}{2 \cdot m} \cdot \Delta P\_L \cdot \left(\frac{R_0 - R_L}{R_L^{-3 \cdot 0.85} - R_0^{-3 \cdot 0.85}} \right) \right]^{\frac{1}{0.85}} = 0.175 \frac{kg}{s}$$

$$w = \frac{\pi(\mathscr{P}_0 - \mathscr{P}_L)R_0^4 \rho}{8\mu L}\left[1 - \frac{1 + (R_L/R_0) + (R_L/R_0)^2 - 3(R_L/R_0)^3}{1 + (R_L/R_0) + (R_L/R_0)^2}\right] \quad \text{(2B.10-3)}$$

$$w_\mu := \frac{\pi \cdot \rho \cdot \Delta P\_L \cdot R_0^4}{8 \cdot \mu} \cdot \left[1 - \frac{1 + \frac{R_L}{R_0} + \left(\frac{R_L}{R_0}\right)^2 - 3 \cdot \left(\frac{R_L}{R_0}\right)^3}{1 + \frac{R_L}{R_0} + \left(\frac{R_L}{R_0}\right)^2} \right] = 0.222 \frac{kg}{s}$$

Problem 2. A piece of foodstuff shaped as a long cylinder with radius R and constant properties (k, $\rho\hat{C}_P$) was taken from a fridge with an initial uniform temperature T_0, and it was heated by hot air at temperature T_∞, flowing orthogonally to the cylinder axis, with velocity v_∞. Under these conditions, the Churchill & Bernstein correlation holds, and the interphase heat transfer coefficient has the value h. After a time t^*, the axial temperature of the foodstuff was measured to be T_A. Calculate:

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The Churchill and Bernstein correlation is (D being the cylinder diameter):

$$N_{Nu} = \frac{hD}{k} = 0.3 + \frac{0.62 N_{Re}^{0.5} N_{Pr}^{0.33}}{\left[1 + \left(\frac{0.4}{N_{Pr}}\right)^{0.67}\right]^{0.25}} \left[1 + \left(\frac{N_{Re}}{282000}\right)^{0.625}\right]^{0.8}$$

The air properties could be taken as constants on their initial values.

Data. $R = 5$ cm, $\rho\hat{C}_P = 6.0$ MJ/(m³K), $T_0 = 5°C$, $T_\infty = 150°C$, $h = 35$ W/(m²K), $t^* = 0.5$ hr, $T_A = 30°C$.

$T_{inf} := 150\,°C \qquad T_0 := 5\,°C \qquad T_A := 30\,°C \qquad R := 5 \cdot cm \qquad h := 35 \cdot \frac{W}{m^2 \cdot K} \qquad t := \frac{1}{2} \cdot hr$

$$\rho C_p := 3000 \cdot \frac{kg}{m^3} \cdot 2000 \cdot \frac{J}{kg \cdot K} = 6 \times 10^6 \cdot \frac{J}{m^3 \cdot K}$$

$$N_{Nu}(N_{Re}, N_{Pr}) := 0.3 + \frac{0.62 \cdot N_{Re}^{0.5} \cdot N_{Pr}^{0.33}}{\left[1 + \left(\frac{0.4}{N_{Pr}}\right)^{0.67}\right]^{0.25}} \cdot \left[1 + \left(\frac{N_{Re}}{282000}\right)^{0.625}\right]^{0.8}$$

1. The Churchill & Bernstein correlation has to be solved iteratively in the unknown v.inf

$$T_f := \frac{T_{inf} + T_0}{2} = 77.5 \cdot °C \qquad N_{Pr.A}(T_f) = 0.706 \qquad k_A(T_f) = 0.03 \cdot \frac{W}{m \cdot K} \qquad \nu_A(T_f) = 2.068 \times 10^{-5} \frac{m^2}{s}$$

$$v_{inf} := 1 \cdot \frac{m}{s} \qquad \text{Given} \qquad \frac{h \cdot 2 \cdot R}{k_A(T_f)} = N_{Nu}\left(\frac{v_{inf} \cdot 2 \cdot R}{\nu_A(T_f)}, N_{Pr.A}(T_f)\right) \qquad v_{inf} := \text{Minerr}(v_{inf}) = 8.052 \frac{m}{s}$$

$$N_{Re} := \frac{v_{inf} \cdot 2 \cdot R}{\nu_A(T_f)} = 3.893 \times 10^4 \qquad N_{Nu}(N_{Re}, N_{Pr.A}(T_f)) = 117.67$$

$$j_H\left(\frac{v_{inf} \cdot 2 \cdot R}{\nu_A(T_f)}, N_{Pr.A}(T_f)\right) = 3.391 \times 10^{-3}$$

2. The transient heating of a cylinder can be solved under the single-term approximation (to be checked later)

$$\theta_A := \frac{T_{inf} - T_A}{T_{inf} - T_0} = 0.828$$

(1) $$\lambda_1 \cdot \frac{J_1(\lambda_1)}{J_0(\lambda_1)} = \frac{h \cdot R}{k}$$

(2) $$A_1 = \frac{2}{\lambda_1} \cdot \frac{J_1(\lambda_1)}{J_1(\lambda_1)^2 + J_0(\lambda_1)^2}$$

(3) $$\theta_A = \theta(t, r = 0) = A_1 \cdot J_0(\lambda_1 \cdot 0) \cdot \exp(-\lambda_1^2 \cdot \tau) = A_1 \cdot J_0(\lambda_1 \cdot 0) \cdot \exp\left(-\lambda_1^2 \cdot \frac{k}{\rho C_P} \cdot \frac{t}{R^2}\right)$$

from (1) $$k = \frac{h \cdot R \cdot J_0(\lambda_1)}{\lambda_1 \cdot J_1(\lambda_1)}$$ (1')

substituting (1') and (2) in (3), one equation (3') in one unknown (λ.1) is obtained

$$\theta_A = \frac{2}{\lambda_1} \cdot \frac{J_1(\lambda_1)}{J_1(\lambda_1)^2 + J_0(\lambda_1)^2} \cdot J_0(\lambda_1 \cdot 0) \cdot \exp\left(-\lambda_1^2 \cdot \frac{h \cdot R \cdot J_0(\lambda_1)}{\lambda_1 \cdot J_1(\lambda_1)} \cdot \frac{1}{\rho C_P} \cdot \frac{t}{R^2}\right)$$ (3')

$$\lambda_1 := 1 \qquad \text{Given} \qquad \theta_A = \frac{2}{\lambda_1} \cdot \frac{J1(\lambda_1)}{J1(\lambda_1)^2 + J0(\lambda_1)^2} \cdot J0(\lambda_1 \cdot 0) \cdot \exp\left(-\lambda_1^2 \cdot \frac{h \cdot R \cdot J0(\lambda_1)}{\lambda_1 \cdot J1(\lambda_1)} \cdot \frac{1}{\rho C_P} \cdot \frac{t}{R^2}\right)$$

$$\lambda_1 := \text{Minerr}(\lambda_1) = 1.144$$

Then, the thermal conductivity could be calculated by eq. (1')

$$k := \frac{h \cdot R \cdot J0(\lambda_1)}{\lambda_1 \cdot J1(\lambda_1)} = 2.209 \cdot \frac{W}{m \cdot K}$$

Biot's number $$N_{Bi} := \frac{h \cdot R}{k} = 0.792$$

Fourier's number $$\tau := \frac{k}{\rho C_P} \cdot \frac{t}{R^2} = 0.265$$ being higher than 0.2, the single-term approximation holds.

3. To calculate the surface temperature, apply the single-term approximation for a long cylinder at the surface (ξ = 1)

$$\theta_S := \frac{2}{\lambda_1} \cdot \frac{J1(\lambda_1)}{J1(\lambda_1)^2 + J0(\lambda_1)^2} \cdot J0(\lambda_1 \cdot 1) \cdot \exp(-\lambda_1^2 \cdot \tau) = 0.578$$

$$T_S := T_{inf} + (T_0 - T_{inf}) \cdot \theta_S = 66.193 \cdot °C$$